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ENTIRE VECTORS AND HOLOMORPHIC EXTENSION OF REPRESENTATIONS. II

BY

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ABSTRACT. Let G be a connected, simply connected Lie group and let G_c be its complexification. Let U be a unitary representation of G. The space of vectors v at which U is holomorphically extendible to G_c is denoted $\mathcal{H}_\infty^{\omega}(U)$. In [9] we characterized those U for which $\mathcal{H}_\infty^{\omega}$ is dense. In the present work we study $\mathcal{H}_\infty^{\omega}$ as a topological vector space, proving e.g., that $\mathcal{H}_\infty^{\omega}$ is a Montel space if U is irreducible and G is nilpotent. We prove a representation theorem for $(\mathcal{H}_\infty^{\omega})'$ which yields a Bergman kernel type theorem for G. As an application we give a necessary and sufficient condition for the set of holomorphic functions on certain solvmanifolds to separate points.

Introduction. In a previous paper [9] we showed that if U is a representation of a type R solvable Lie group G in a Banach space B, then there is a dense set of vectors v for which $g \to U_g v$ is extendible to a holomorphic map of the complexification G_c of G into B. Such vectors are called entire vectors and the space of such vectors is denoted by $\mathcal{H}_\infty^{\omega}(U)$. $\mathcal{H}_\infty^{\omega}$ is an invariant subspace and the restriction U^{ω} of U to $\mathcal{H}_\infty^{\omega}$ possesses a holomorphic extension to G_c.

One may put on $\mathcal{H}_\infty^{\omega}$ a Fréchet space structure defined via the family of norms

$$\rho_{\Omega}(v) = \sup_{z \in \Omega} \|U_z^{\omega} v\|$$

where $\Omega \subset G_c$ and is compact.

We refer to the above topology as the $\mathcal{H}_\infty^{\omega}$ topology. Under this topology U^{ω} is a continuous (in fact, holomorphic) representation of G_c by continuous operators. (For the proofs of these facts see §I.)

In this paper we propose to study U^{ω} via the topological structure of $\mathcal{H}_\infty^{\omega}$. Our main interest is in the Hilbert space-unitary case.

In this case we prove that $\mathcal{H}_\infty^{\omega}$ is reflexive and, if B is separable, so is $\mathcal{H}_\infty^{\omega}$. We also show that under a certain closely related, but weaker, topology $\mathcal{H}_\infty^{\omega}$ is a semi-Montel space. It turns out that if G is nilpotent and U is irreducible, the weaker topology in fact equals the $\mathcal{H}_\infty^{\omega}$ topology and $\mathcal{H}_\infty^{\omega}$ is a Montel space.

Our main tool is a representation theorem for the dual space $(\mathcal{H}_\infty^{\omega})'$ which states (in the unitary case) that every continuous linear functional is a *finite*

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linear combination of functionals of the form $v \to (U_z^{\omega} v, w)$ ($z \in G_c$, $w \in B$ fixed).

In a large number of cases (e.g. when U is an induced representation) it can be shown that $\mathcal{H}_\infty^\omega$ can be identified with a space of entire functions on G_c for which point evaluation is continuous. Our representation theorem yields a Bergman kernel type theorem for certain unbounded domains.

As an application of our results, we develop a necessary and sufficient condition for the entire functions on the complexification of a compact solvmanifold to separate points of the solvmanifold.

I. Throughout the sequel G will denote a connected, simply connected, solvable, type R Lie group and $\mathfrak{L}$ will be its Lie algebra. Let U be a fixed unitary representation of G in a Hilbert space $\mathcal{H}$. Let $C^\infty(U)$ be the space of C^∞ vectors for U and, for $X \in \mathfrak{L}$ and $v \in C^\infty(U)$, let

$$\partial U(X)v = \lim_{t \to 0} ((U_{\exp tX} v - v)/t).$$

The mapping $X \to \partial U(X)$ defines a representation of $\mathfrak{L}$ by skew-symmetric operators which we extend to a representation of the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{L}$.

Let $X = (X_1, \cdots, X_d)$ be a fixed, ordered, Jordan-Hölder basis for G. Let N denote the positive integers. If $n \in N^d$, let $X^n = X_1^{n_1} \cdots X_d^{n_d}$. Also if $n \in N$, let

$$\{X\}^n = \{Y_i \cdots Y_n \mid Y_i \in \{X_1 \cdots X_d\},\ 1 \le i \le n\}.$$

For $v \in C^\infty(U)$ and $t > 0$, let

$$\sigma_t(v) = \sup\{\|\partial U(Y)v\| t^n/n! \mid Y \in \{X\}^n,\ n \in N\}$$

$$\tau_t(v) = \sup\{\|\partial U(X^n)\| t^{|n|}/n! \mid n \in N^d\}.$$

Recall ((I.1) and (I.3) of [9]) that $v \in \mathcal{H}_\infty^\omega$ iff $\sigma_t(v)$ (or equivalently $\tau_t(v)$) is finite for all $t > 0$.

Proposition (I.1). *$\mathcal{H}_\infty^\omega(U)$ is a Fréchet space and U^ω is a holomorphic representation of G_c in $\mathcal{H}_\infty^\omega$ by continuous operators. Furthermore, the topologies on $\mathcal{H}_\infty^\omega$ defined via the families of seminorms $\{\sigma_t\}$ and $\{\tau_t\}$ both agree with the $\mathcal{H}_\infty^\omega$ topology.*

Proof. If K_n is any nested sequence of compact subsets of G_c for which $K_n \subset \mathring{K}_{n+1}$ and $\bigcup_{n=1}^\infty K_n = G_c$, then ρ_{K_n} form a basis of seminorms for $\mathcal{H}_\infty^\omega$. Hence $\mathcal{H}_\infty^\omega$ is countably normed and is a metric space.

If v_n is Cauchy in $\mathcal{H}_\infty^\omega$ then $z \to U_z v_n$ converges (as a map of G_c into $\mathcal{H}$) uniformly on compacta to a holomorphic function $\phi(z)$. It is easily seen that $v_n \to \phi(e)$ in $\mathcal{H}_\infty^\omega$. Thus $\mathcal{H}_\infty^\omega$ is a Fréchet space.

It is easily verified that $\mathcal{H}_\infty^\omega$ is also a Fréchet space in the topologies defined by the $\{\sigma_t\}$ and $\{r_t\}$. The equality of the various topologies then follows from the closed graph theorem and the fact that the injection $\mathcal{H}_\infty^\omega \to \mathcal{H}$ is continuous in each topology.

That U_z^ω is a continuous operator follows easily from the definitions. To show that U^ω is holomorphic in $\mathcal{H}_\infty^\omega$, it suffices, in virtue of (I.1) of [9], to show that $\{\partial U(Y) v t^n/n! \mid Y \in \{X\}^n,\ n \in N\}$ is bounded in $\mathcal{H}_\infty^\omega$ for all $t > 0$ and all $v \in \mathcal{H}_\infty^\omega$. This set, however, is clearly bounded in each σ_s, $s > 0$, and hence in $\mathcal{H}_\infty^\omega$, thus proving the analyticity. Q.E.D.

Remarks. The equality of the topologies could also be shown directly from the estimates in §I of [9]. Note also that we have so far used only the Banach property of $\mathcal{H}$.

Yet another description of the $\mathcal{H}_\infty^\omega$ topology is possible.

Definition (I.2). Let $s \in \mathbf{R}^d$, $s = (s_1, \cdots, s_d)$, $s_i > 0$. Let $\mathcal{H}^s(U) = \mathcal{H}^s = \{v \in \mathcal{H} \mid t \to U_{\exp tX_i} v$ is extendible to a continuous map of $|z| \le s_i$ which is holomorphic on $|z| < s_i$ for $i = 1, \cdots, d\}$. Define

$$(v, w)_s = \sum_{i=1}^{d} \{(U_{\exp is_jX_j} v, U_{\exp is_jX_j} w) + (U_{\exp -is_jX_j} v, U_{\exp -is_jX_j} w)\}.$$

Let $\|v\|_s^2 = (v, v)_s$.

Proposition (I.3). *Under* $\|\ \|_s$, $\mathcal{H}^s$ *is a Hilbert space. Also* $\mathcal{H}_\infty^\omega = \bigcap_{s_i > 0} \mathcal{H}^s$ *and the topology of* $\mathcal{H}_\infty^\omega$ *is given by the family of restrictions of the norms* $\|\ \|_s$.

Proof. From the maximum modulus principle and the unitarity of U, it follows that

$$\sup_{|z| \le s_j} \|U_{\exp zX_j} v\|^2 \le \|U_{\exp is_jX_j} v\|^2 + \|U_{\exp -is_jX_j} v\|^2$$

for $v \in \mathcal{H}^s$ and $j = 1, \cdots, d$. The completeness of $\mathcal{H}^s$ follows from this.

That $\bigcap \mathcal{H}^s = \mathcal{H}_\infty^\omega$ follows from (I.5) of [9]. The topological part is another application of the closed graph theorem. Q.E.D.

The above proposition yields a characterization of $(\mathcal{H}_\infty^\omega)'$. If $s = (s_1, \cdots, s_d) \in \mathbf{R}^d$, $s_i > 0$, let $U_s: \mathcal{H}^{2s} \to \mathcal{H}$ be given by

$$U_s = \sum_{j=1}^{d} \{U_{\exp_i 2s_jX_j} + U_{\exp -i2s_jX_j}\}.$$

Also, if $r, t \in \mathbf{R}^d$, we will say $t > r$ iff $t_i > r_i$ for $i = 1, \cdots, d$.

Theorem (I.4). *Given* $\phi \in (\mathcal{H}_\infty^\omega)'$, *there is an* $s_0 \in \mathbf{R}^d$, $s_0 > 0$, *with the following property: For all* $s > s_0$ *there is a* $w_s \in \mathcal{H}^s$ *which represents* ϕ *in the* *sense that* $\phi(v) = (U_s v, w_s)$ *for all* $v \in \mathcal{H}_\infty^\omega$.

Proof. Note that if $w, v \in \mathcal{H}^s$, then, by uniqueness of analytic continuation, $(U_{\exp zX_j} v, w) = (v, U_{\exp -\bar{z}X_j} w)$ for all $|z| \leq s_j$. It follows that, for $v \in \mathcal{H}_\infty^\omega$ and $w \in \mathcal{H}^s$, $(U_s v, w) = (v, w)_s$. Thus, our assertion is that there is an s_0 for which ϕ is continuous in $\| \|_s$ for all $s > s_0$. This follows from (I.3) and its proof. Q.E.D.

Corollary (I.5). *A subspace* M *of* $\mathcal{H}_\infty^\omega$ *is dense iff there is a sequence* $s^p \in \mathbf{R}^d$ *for which* $s_i^p \to \infty$ *as* $p \to \infty$, $i = 1, \cdots, d$, *and for which* $U_{s^p}(M)$ *is dense in* $U_{s^p}(\mathcal{H}_\infty^\omega)$ *in the topology of* $\mathcal{H}$ *for all* $p \in N$.

Proof. The usual annihilator argument.

Corollary (I.6). *If* $\mathcal{H}$ *is separable, so is* $\mathcal{H}_\infty^\omega$.

Proof. Let v_n, $n = 1, \cdots$, be a countable dense subset of $\mathcal{H}$. For $n, m \in N$, $s \in \mathbf{R}^d$ if there is a point $w \in \mathcal{H}_\infty^\omega$ for which $\|v_m - U_s w\| < 1/n$, let $w_s^{n,m}$ be such a point. Otherwise let $w_s^{n,m} = 0$. Let $W_s = \{w_s^{n,m} \mid n, m \in N\}$ and let $M =$ span $\bigcup_{s \in N^d} W_s$. $U_s (W_s)$ is dense in $U_s(\mathcal{H}_\infty^\omega)$ for $s \in N^d$. Hence M is dense in $\mathcal{H}_\infty^\omega$ by (I.5). Finally, the set of linear combinations with complex rational coefficients of elements of $\bigcup_{s \in N^d} W_s$ is a countable dense subset of $\mathcal{H}_\infty^\omega$. Q.E.D.

Remarks. In view of the proof of Theorem (II.2) of [9], we might expect that one could produce a countable dense set of entire vectors via regularization — i.e. vectors of the form $\int_G \phi(g) U_g v \, dg$ where $\phi(g)$ is an L^1 entire vector. Although this definitely seems possible, we are unable to prove that the set of such vectors is dense in $\mathcal{H}_\infty^\omega$. The difficulty is that the integral does not seem to converge in $\mathcal{H}_\infty^\omega$. It is for this reason, also, that we are unable to prove that U^ω is topologically irreducible if U is (cf. Example (8.20) of [10]).

(I.4) also provides, in the case that U is given as a direct integral of other representations, another description of $(\mathcal{H}_\infty^\omega)'$.

This description is based on the following fundamental theorem of Goodman's [4, Lemma (3.1)].

Theorem (A.1). *If* $U = \int_M \bigoplus U^\alpha \, d\alpha$ *where* M *is an analytic Borel space and* U^α *is an integrable family of unitary representations of* G *in* $\mathcal{H}^\alpha$, *then* $v \in \mathcal{H}$, $v = \{v^\alpha\}$ *is in* $\mathcal{H}_\infty^\omega(U)$ *iff* $v^\alpha \in \mathcal{H}_\infty^\omega(U^\alpha)$ *for a.e.* α *and, for all compact sets* $\Omega \subset G_c$, $\alpha \to \sup_{z \in \Omega} \|U_z^\alpha v^\alpha\|^2$ *is in* $L^1(M)$.

In this case, $(U_z v)^\alpha = U_z^\alpha v^\alpha$ for all $z \in G_c$ and a.e. α.

If the representation space of U is separable then the same statement, more or less, is true for $(\mathcal{H}_\infty^\omega)'$.

Proposition (I.7). *In the notation of* (A.1), *if* $\phi \in \mathcal{H}^{\omega}_{\infty}(U)'$ *then for a.e.* α *there is a uniquely determined functional* $\phi^{\alpha} \in \mathcal{H}^{\omega}_{\infty}(U^{\alpha})'$ *for which* $\phi(v) = \int_M \phi^{\alpha}(v^{\alpha})\, d\alpha$ *for all* $v = \{v^{\alpha}\} \in \mathcal{H}^{\omega}_{\infty}(U)$. *The integral is absolutely convergent.*

Conversely, if $\phi^{\alpha} \in \mathcal{H}^{\omega}_{\infty}(U^{\alpha})'$ *are such that* $\alpha \to \phi^{\alpha}(v^{\alpha})$ *is in* $L^1(M)$ *for all* $v = \{v^{\alpha}\} \in \mathcal{H}^{\omega}_{\infty}(U)$, *then* $v \to \int_M \phi^{\alpha}(v^{\alpha})\, d\alpha$ *defines an element of* $\mathcal{H}^{\omega}_{\infty}(U)'$.

Proof. The first part, except for uniqueness, follows from (I.4) and (A.1). The uniqueness follows as in (C.1) of [8] except that we now obtain the required countable dense subset of $\mathcal{H}^{\omega}_{\infty}(U^{\alpha})$ via the following lemma.

Lemma. *If* $\{v_n\}_{n=1}^{\infty}$ *is dense in* $\mathcal{H}^{\omega}_{\infty}(U)$, *then for a.e.* $\alpha \in M$, $\{v_n^{\alpha}\}_{n=1}^{\infty}$ *is dense in* $\mathcal{H}^{\omega}_{\infty}(U^{\alpha})$.

Proof. Let $s^p \in \mathbf{R}^d$, $p \in N$, be a sequence satisfying the hypothesis of (1.5). Let $K^p = U_{s^p}(\mathcal{H}^{\omega}_{\infty}(U))$ and let $K^p_{\alpha} = U^{\alpha}_{s^p}(\mathcal{H}^{\omega}_{\infty}(U^{\alpha}))$. If $E \subset M$ is measurable, let $\Pi_E\colon \mathcal{H} \to \mathcal{H}$ be the map that takes $\{v^{\alpha}\}$ onto $\{w^{\alpha}\}$ where $w^{\alpha} = v^{\alpha}$ if $\alpha \in E$ and is zero otherwise. It follows from (A.1) that Π_E leaves K^p invariant. Hence Π_E commutes with the projection Π^p onto K^p. It follows from Theorem (P. 6) of [6, p. 92] that Π^p is a direct integral of projections Π^p_{α}. Π^p_{α} is the projection onto K^p_{α} and, hence K^p is the direct integral of the K^p_{α}.

But $U_{s^p}(\{v_n\}_{n=1}^{\infty})$ is dense in K^p. Thus $U^{\alpha}_{s^p}(\{v^{\alpha}_n\}_{n=1})$ is, for a.e. α, dense in K^p_{α}. Upon choosing a set of α for which this is true for all $p \in N$, the lemma follows from (I.5).

The converse statement of (I.7) follows from the closed graph theorem as in [8]. Q.E.D.

Corollary (I.8). *If in* (I.7) *each* U^{α} *is finite dimensional, then* ϕ *is given via a function* $\alpha \to w^{\alpha} \in \mathcal{H}^{\alpha}$ *in the sense that* $\phi(v) = \int_M (v^{\alpha}, w^{\alpha})\, d\alpha$ *for all* $v \in \mathcal{H}^{\omega}_{\infty}(U)$.

Furthermore $\alpha \to w^{\alpha}$ *represents an element of* $\mathcal{H}^{\omega}_{\infty}(U)'$ *iff* $\alpha \to (v^{\alpha}, w^{\alpha})$ *is integrable for all* $\{v^{\alpha}\} \in \mathcal{H}^{\omega}_{\infty}(U)$.

Example (I.9). Let $G = \mathbf{R}$ and let U be the regular representation. $\mathbf{R}_c$ is $\mathbf{C}$ and $\mathcal{H}^{\omega}_{\infty}(U)$ is the space of entire functions f on $\mathbf{C}$ for which

$$\sup_{|y| \le \delta} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx < \infty$$

for all $\delta > 0$ (cf. [3, p. 64]).

By the Paley-Wiener theorem this is the space of functions which, when restricted to the real line, satisfy $\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \exp\{2\delta|\lambda|\}\, d\lambda < \infty$ for all

$\delta > 0$. ($\hat{}$ indicates Fourier transform.)

$\hat{}$ defines a direct integral decomposition of U into one dimensional representations. (I.8) applies and shows that, to each element ϕ of $(\mathcal{H}_\infty^\omega)'$, there is a unique function $\hat{\phi}$: $\mathbf{R} \to \mathbf{C}$ for which $\phi(f) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\hat{\phi}(\lambda)\, d\lambda$. The space of such $\hat{\phi}$ can be characterized as the set of functions for which $|\hat{\phi}(\lambda)| \leq e^{\delta|\lambda|}$ for some $\delta > 0$ (depending on ϕ).

One of our main uses of (I.3) will be in the study of the duality theory of $\mathcal{H}_\infty^\omega$.

Definition (I.10). For $w \in \mathcal{H}_\infty^\omega$ and $z \in G_c$ define the seminorm $\| \; \|_{z,w}$ by $\|v\|_{z,w} = |(U_z v, w)|$ for $v \in \mathcal{H}_\infty^\omega$. Let $\mathfrak{W}_\infty^\omega = \mathfrak{W}_\infty^\omega(U)$ be $\mathcal{H}_\infty^\omega$ with the topology defined by the family $\|\cdot\|_{z,w}$.

Proposition (I.11). *$\mathfrak{W}_\infty^\omega$ is $\mathcal{H}_\infty^\omega$ with its weak topology.*

Proof. (I.4).

We shall also need an intermediary topology.

Definition (I.12). If $\Omega \subset G_c$, Ω compact, let $\|v\|_{\Omega,w} = \sup_{z\in\Omega} \|v\|_{z,w}$ for $v, w \in \mathcal{H}_\infty^\omega$. Let $\mathfrak{M}_\infty^\omega$ be $\mathcal{H}_\infty^\omega$ with the topology defined via the family $\|\cdot\|_{\Omega,w}$.

Recall that a locally convex topological vector space $\mathfrak{F}$ is said to be semi-Montel if all closed and bounded sets are compact. If $\mathfrak{F}$ is also barrelled, $\mathfrak{F}$ is said to be Montel.

Proposition (I.13). *$\mathfrak{M}_\infty^\omega$ and $\mathfrak{W}_\infty^\omega$ are semi-Montel spaces.*

Proof. By the uniform boundedness principle in $\mathcal{H}_\infty^\omega$, $\mathfrak{M}_\infty^\omega$, $\mathcal{H}_\infty^\omega$, and $\mathfrak{W}_\infty^\omega$ all have the same bounded sets. Since the injection $\mathfrak{M}_\infty^\omega \to \mathfrak{W}_\infty^\omega$ is continuous, it suffices to prove the assertion for $\mathfrak{M}_\infty^\omega$.

Let $\{v_\alpha\}_{\alpha\in A}$ be a bounded net in $\mathfrak{M}_\infty^\omega$. Then $\{v_\alpha\}_{\alpha\in A}$ is bounded in $\mathcal{H}$. Hence we may assume v_α converges weakly to some element $v \in \mathcal{H}$. Now, let $w \in \mathcal{H}$ and set $\phi_w(z) = \sup_{\alpha\in A} |(U_z v^\alpha, w)|$, $z \in G_c$. Let Ω_w be the space of entire functions on G_c which are dominated by ϕ_w. From the boundedness of v^α, ϕ_w is bounded on compact subsets. Hence Ω_w is a closed bounded subset of the space of entire functions with the compact open topology. By Montel's theorem Ω_w is compact. Let $\Omega = \Pi_{w\in\mathcal{H}} \Omega_w$. By the Tychonoff theorem Ω is compact. Let $\{X_\alpha\}_{\alpha\in A} \in \Omega$ be the net $X_\alpha^w = z \to (U_z v^\alpha, w)$ and let $\{X_{n_\beta}\}_{\beta\in A'}$ be a convergent subnet, say $X_{n_\beta} \to X$. Then $X_{n_\beta}^w \to X^w$ uniformly on compact subsets of G_c for all $w \in \mathcal{H}$. X^w is an entire function and, for $g \in G$, $X^w(g) = \lim_{\beta\in A'} (U_g v_{n_\beta}, w) = (U_g v, w)$. Thus, by (I.2) of [9], $v \in \mathcal{H}_\infty^\omega$ and $v_{n_\beta} \to v$ in $\mathfrak{M}_\infty^\omega$. Q.E.D.

Corollary (I.14). *$\mathcal{H}_\infty^\omega$ is reflexive.*

Proof. Since $\mathfrak{W}_\infty^\omega$ is semi-Montel, $\mathcal{H}_\infty^\omega$ is weakly reflexive by Proposition 1 of [5, p. 227]. $\mathcal{H}_\infty^\omega$ is barrelled since it is Fréchet. Our conclusion then follows from Proposition 6 of [5, p. 229]. Q.E.D.

Remark. The above proposition is actually true for any representation of G in a reflexive Fréchet space. The same proof carries over with only minor changes.

Corollary (I.15). *The relative topologies gotten by restricting $\mathfrak{M}_\infty^\omega$ and $\mathfrak{W}_\infty^\omega$ to a fixed bounded set agree. In particular a sequence x_n converges in $\mathfrak{M}_\infty^\omega$ iff it converges in $\mathfrak{W}_\infty^\omega$.*

Proof. It suffices to consider only the $\mathfrak{M}_\infty^\omega$ bounded sets, in which case the proposition follows from compactness. Q.E.D.

In general, the topology of $\mathcal{H}_\infty^\omega$ cannot be any nicer than that of $\mathcal{H}$ since if U is uniformly continuous, for example, $\mathcal{H}_\infty^\omega = \mathcal{H}$. However, if G is nilpotent and U is irreducible, the topology is, in some sense, significantly nicer.

Theorem (I.16). *If G is nilpotent and U is irreducible, then $\mathcal{H}_\infty^\omega$ is a Montel space.*

Before proving (I.16), we comment that we do not know the most general type of group G for which the theorem is true. However (I.16) implies that G is C.C.R. for, if f is any entire vector for the L^1 left regular representation of G, then $U_f = \int_G f(g)\, Ug\, dg$ maps $\mathcal{H}$ into $\mathcal{H}_\infty^\omega$. By the closed graph theorem U_f is continuous. From the continuity of the injection $\mathcal{H}_\infty^\omega \to \mathcal{H}$, it follows that U_f, as a map of $\mathcal{H}$ into $\mathcal{H}$, is a compact operator if $\mathcal{H}_\infty^\omega$ is a Montel space. Finally, every operator U_h, $h \in L^1(G)$, is a uniform limit of such U_f. Hence every such U_h is compact and G is C.C.R.

Auslander-Moore [1, Chapter V] have shown that, for type R groups, C.C.R. is equivalent to type I. We suspect that (I.16) is also.

Proof of (I.16). Let B be closed and bounded in $\mathcal{H}_\infty^\omega$. To show that B is compact it suffices, we claim, to show that $U_z B$ is compact in $\mathcal{H}$ for all $z \in G_c$, for in this case $\mathfrak{B} = \Pi_{z \in G_c} U_z B$ is compact. Hence every net $\{x_\alpha\}_{\alpha \in A}$ in B has a subnet $\{y_\beta\}_{\beta \in A'}$ for which $U_z y_\beta$ converges in $\mathcal{H}$ for all $z \in G_c$. By (I.3) y_β then converges in $\mathcal{H}_\infty^\omega$. Thus, it suffices to show that every $\mathcal{H}_\infty^\omega$ bounded set B is compact in $\mathcal{H}$.

Now in [12], Pukánszky showed (Part II, Chapter II, Theorem 2 and its proof) that since U is irreducible, there is an element $X \in \mathfrak{U}(G)$ for which $\partial U(X)$ has a bounded inverse T. From the eigenvalues for T as computed in [12], T is a compact operator. Hence $B = T\,\partial U(X)B$ is a compact subset of $\mathcal{H}$. Q.E.D.

II. **Function spaces.** We are, in this section, interested in representations realized in spaces of functions on G as defined below.

Definition (II.1). Let $\mathcal{H}_0$ be a Hilbert space and let $\mathcal{H}$ be a space of $\mathcal{H}_0$ valued locally integrable functions (with a.e. equal functions identified) with respect to Haar measure. Suppose $\mathcal{H}$ is topologized in such a manner that:

(1) $\mathcal{H}$ is a Banach space.

(2) The injection of $\mathcal{H}$ into $L^1_{\mathrm{loc}}(G, \mathcal{H}_0)$ is continuous.

(3) For all $f \in \mathcal{H}$ and $g \in G$, the function $R_g f: x \to f(xg)$ is in $\mathcal{H}$ and $g \to R_g$ defines a continuous representation of G in $\mathcal{H}$.

Then $\mathcal{H}$ is said to be a *regular Banach space of* $\mathcal{H}_0$*-valued functions* and R is said to be the regular representation of G in $\mathcal{H}$.

If $\mathcal{F}$ is any Fréchet space and $F: G \to \mathcal{F}$ is a C^∞ function then define, for $X \in \mathfrak{L}$,

$$\tilde{X}F(g) = \lim_{t \to 0} \frac{F(g \exp tX) - F(g)}{t}.$$

In this terminology, we have the following generalization of results of Goodman [4] and Poulsen [11].

Proposition (II.2). *Let $f \in \mathcal{H}$. $f \in C^\infty(R)$ iff f is a C^∞ $\mathcal{H}_0$valued function and $\tilde{X}_1 \cdots \tilde{X}_n f \in \mathcal{H}$ for all $X_1, \cdots, X_n \in \mathfrak{L}$.*

$f \in \mathcal{H}^{\omega}_{\infty}(R)$ iff f is extendible to an entire $\mathcal{H}_0$ valued map of G_c for which the maps $R_z f: g \to f(gz)$ are in $\mathcal{H}$ for all $z \in G_c$ and $z \to R_z f$ is continuous in $\mathcal{H}$.

Proof. Suppose f satisfies the hypothesis of the "only if" part of the above. For each $t \in \mathbf{R}$ and $X \in \mathfrak{L}(G)$ let $g_t = \int_0^t R(\exp tX)\tilde{X}f\, dt$. (This $\mathcal{H}$ valued integral exists by Lemma 2, p. 12 of [7].) Let $w \in (\mathcal{H}_0)'$ and let $\phi \in \mathfrak{D}(G)$. By evaluating with functionals of the form

$$g \to \int_G \phi(x) \langle g(x), w \rangle \, dx \qquad (g \in \mathcal{H})$$

it is easily seen that $g_t = R(\exp tX)f - f$. It follows that f is a C^1-vector and hence, by induction, that f is a C^∞-vector. Conversely, if f is a C^∞-vector for R and $w \in (\mathcal{H}_0)'$, then it follows by evaluating $\partial R(X)f$ with functionals of the above form that $x \to \langle \partial R(X)f(x), w \rangle$ is the distributional derivative (along X) of $x \to \langle f(x), w \rangle$. It follows by induction and the Sobolev theorem that $x \to \langle f(x), w \rangle$ is a C^∞-function. Since weak C^∞ implies strong C^∞, we get that f is a C^∞ $\mathcal{H}_0$valued map (see [7, Lemma 1, p. 47]).

To prove the entirety part, let $f \in \mathcal{H}^{\omega}_{\infty}(R)$. Then f is, in particular, a C^∞-vector and hence is continuous. Let K be a compact neighborhood of e in G and let $\mathcal{C}(K, \mathcal{H}_0)$ be the Banach space of continuous $\mathcal{H}_0$ valued functions on

K given the sup-norm topology.

The restriction map $f \to f|_K$ of $\mathcal{H}_\infty^\omega(R)$ into $\mathcal{C}(K, \mathcal{H}_0)$ is continuous by the closed graph theorem and the regularity of $\mathcal{H}$. It follows that point evaluation is a continuous linear transform on $\mathcal{H}_\infty^\omega(R)$. We conclude from (I.1) above that $z \to (U_z^\omega f)(e)$ is a holomorphic function on G_c. This is the desired extension. Clearly U^ω acts as claimed.

Finally, if f has an extension satisfying the above hypothesis, let γ be a closed curve in G_c. By evaluating with the appropriate functionals (as in the C^∞ part), it is easily shown that $\int_\gamma R_z f\, dz \in \mathcal{H}$ is zero. Hence Morera's theorem finishes the proof.

Corollary (II.3). *Let R be the unitary representation of G induced from a unitary representation L of a closed subgroup of G. If U is realized as in Blattner* [2], *then R is a regular representation and the characterization of* (II.2) *applies.*

Corollary (II.4). *Let $\mathcal{H}_0 = \mathbf{C}$, $\mathcal{H} = L^p(G)$ $(1 \le p \le \infty)$. Let R be the right regular representation of G in $\mathcal{H}$. Then* (II.2) *applies and we obtain the expected characterization of $\mathcal{H}_\infty^\omega(R)$.*

Remark (II.2) is a generalization of results from [11]. (II.4) in the nilpotent case is due to Goodman [4].

Remark (II.5). Note that in the above proof we showed that $f \to f(e)$ is continuous from $\mathcal{H}_\infty^\omega(R)$ into $\mathcal{H}_0$. It follows that $f \to f(z)$ is continuous for all $z \in G_c$. If $\mathcal{H}_0 = \mathbf{C}$ we obtain from (I.4) a Cauchy-like representation theorem for $\mathcal{H}_\infty^\omega(R)$. In fact, if R is the right regular representation of G, our representation theorem takes the following form:

Corollary. *Let R be the L^2 right regular representation. Then, for all $z \in G_c$, there is an $s_0 \in \mathbf{R}^d$, $s_0 > 0$, with the following property: For all $s > s_0$ there is a $w_s^z \in \mathcal{H}^s(R)$ such that*

$$f(z) = \sum_j \left\{ \int_G f(g \exp is_j X_j) \overline{w}_s^z(g)\, dg + \int_G f(g \exp - is_j X_j) \overline{w}_s^z(g)\, dg \right\}$$

for all $f \in \mathcal{H}_\infty^\omega(R)$,

Note, incidentally, that the modular function does not appear in the above formula. The reason is that for type R groups, the trace of the adjoint representation is unity and hence G is unimodular. Type R groups are the only ones whose regular representation has nonzero entire vectors.

In general, the kernels w_s^z seem to be difficult to compute. For R the right regular representation and z real, it suffices to compute w_s^e, for let L be the L^2 left regular representation of G. Since L and R commute, L leaves $\mathcal{H}^s(R)$ invariant and L is unitary in $\| \; \|_s$. Thus, for $f \in \mathcal{H}_\infty^\omega(R)$ and $g \in G$,

$$f(g) = L_{g^{-1}} f(e) = (L_{g^{-1}} f, w_s^e)_s = (f, L_g w_s^e)_s.$$

Thus $w_s^g = L_g w_s^e$.

If $G = \mathbf{R}$, we may explicitly compute $w_{s/2}^e = w$ as follows: From the Paley-Wiener theorem and (II.2), it follows that the Fourier transform $\hat{f}$ of a function $f \in L^2(G)$ is in $\mathcal{H}_\infty^\omega(R)$ iff $\int_{-\infty}^\infty e^{a|t|} |f(t)|^2 dt < \infty$ for all $a \geq 0$. (Cf. Goodman [3, p. 64], for details.) The analytic extension of $\hat{f}$ is given by

$$\hat{f}(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-izx} f(x)\, dx.$$

Hence, letting $\smile$ denote the conjugate of the inverse Fourier transform

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)\, dx = \hat{f}(0) = \int_{-\infty}^{\infty} \hat{f}(x + is)\overline{w}(x)\, dx + \int_{-\infty}^{\infty} \hat{f}(x - is)\overline{w}(x)\, dx$$

$$= \int_{-\infty}^{\infty} 2(\cosh sx) f(x) \breve{w}(x)\, dx.$$

Thus $\breve{w}(x) = 2(2\pi)^{-1/2}/\cosh sx$. The Fourier transform can be explicitly computed via a contour integral to be $w(x) = [4s \cosh(\pi x/2s)]^{-1}$.

III. Applications to complex solvmanifolds. Let S be a solvable, connected, simply connected Lie group (not necessarily type R) and let Γ be a closed but not necessarily connected subgroup. Then the homogeneous space $M = S/\Gamma$ is a solvmanifold. If S is a complex Lie group and Γ is a complex Lie subgroup (i.e. the component of the identity of Γ is a complex analytic subgroup of S), then S/Γ is a complex manifold and will be called a complex solvmanifold.

Lemma (III.1). *Let Γ_0 be the component of the identity of Γ and let $(\Gamma_0)_c \subset S_c$ be its complexification. Let $\Gamma_c = \Gamma \cdot (\Gamma_0)_c$. Then Γ_c is a closed, complex Lie subgroup of S_c and M is canonically imbedded in $M_c = S_c/\Gamma_c$.*

Proof. There is a Jordan-Hölder basis B of $\mathfrak{L}(S)$ which contains a Jordan-Hölder basis B_0 of $\mathfrak{L}(\Gamma_0)$. Letting B define holomorphic coordinates for S_c as in the proof of (I.1) of [9], one sees that B_0 defines holomorphic coordinates for $(\Gamma_0)_c$ and hence $(\Gamma_0)_c \cap S = \Gamma_0$. Let $\Gamma_c = \Gamma \cdot (\Gamma_0)_c$.

We claim that $(\Gamma_0)_c$ is invariant under Γ and hence that Γ_c is a subgroup. To see this let $\| \;\|$ be a complex norm on $\mathfrak{L}_c(S)$ and let $\mathfrak{U}$ be a closed ball in $\mathfrak{L}_c(S)$ such that

(i) exp is a homeomorphism onto the image U of $\mathfrak{U}$ in S_c, and

(ii) $\mathfrak{U}$ is sufficiently small in the sense defined below.

For $z \in S_c$ and $\alpha \in \mathbf{C}$, $|\alpha| \leq 1$, define $\alpha z = \exp(\alpha(\log z))$ wherever $\log z$ is defined and single valued. If $z \in U$ and U is sufficiently small, the elements $y = (-i/2)(\overline{z}^{-1} z)$ and $x = z(iy)^{-1}$ are defined and satisfy $z = x(iy)$ ($-$ denotes the canonical conjugation on S_c). Furthermore, it is easily seen that $y = \overline{y}$, $x = \overline{x}$ and, hence, that y and x are in $(\Gamma_0)_c \cap S = \Gamma_0$. It follows that if g is a fixed element of Γ and U is small enough (relative to g), then

$$gzg^{-1} = (gxg^{-1})(g(iy)g^{-1}) = (gxg^{-1})(i(gyg^{-1})) \in (\Gamma_0)_c.$$

Since $U \cap (\Gamma_0)_c$ generates $(\Gamma_0)_c$, g leaves $(\Gamma_0)_c$ invariant. Hence $\Gamma \cdot (\Gamma_0)_c$ is a subgroup.

It follows similarly that Γ_c is closed since to show closure it suffices to show that there is a closed neighborhood U of e in S_c such that $U \cap \Gamma$ is closed. If U is sufficiently small we can work with real and imaginary parts as above.

The map $S/\Gamma \to S \cdot \Gamma_c/\Gamma_c \subset M_c$ is easily seen to be an imbedding (since $\Gamma_c \cap S = \Gamma$) and the lemma follows. Q.E.D.

We are interested in the following questions about M_c.

(i) Under what conditions does the set of entire functions $\Omega(M_c)$ separate points of M?

(ii) When does $\Omega(M_c)$ separate points of M_c?

Our answers are as follows:

Theorem (III.2). *Supopse Γ contains no nontrivial normal analytic subgroups. Then*:

(i) *If S is type R, then $\Omega(M_c)$ separates points of M. If S/Γ is compact, it is also necessary that S be type R for point separating to hold.*

(ii) *If S is type R, then there is a closed complex subgroup $\Gamma' \subset S_c$ with the following properties*:

(a) $\Gamma' \supset \Gamma_c$ *and* $(\Gamma')_0 = (\Gamma_c)_0 = (\Gamma_0)_c$.

(b) $\Gamma' \cap S = \Gamma$ *and there is a natural imbedding of M into S_c/Γ'.*

(c) $\Omega(S_c/\Gamma')$ *separates points of* S_c/Γ'.

I.e., if we change our notion of complexification slightly, M has a complexification for which the holomorphic functions separate points.

Proof. (i) Let S be type R. Let $C_0(M)$ be the Banach space all continuous functions vanishing at infinity given the sup norm. We may identify $C_0(M)$ with a space of functions $\mathcal{C}$ on G which are invariant under right translation by elements of Γ. S acts on $\mathcal{C}$ via left translation and this action defines a representation R of S in $\mathcal{C}$. $\mathcal{C}$ is a regular Banach space of complex functions (in the sense of II.1) and R is the left regular representation of S in $\mathcal{C}$. By (II.3) of [9], R has a dense set of entire vectors and by (II.2) above each entire vector f extends to an entire function f_c on S_c. Since f is invariant under Γ, f_c is invariant under $(\Gamma_0)_c$ (by uniqueness of analytic extensions in $(\Gamma_0)_c$) and hence under Γ_c. Thus, upon projection f_c defines an element $\tilde{f}_c$ of $\Omega(M_c)$. The set of such $\tilde{f}_c$ separate points of M since they are dense in $C_0(M)$.

If S/Γ is compact, but not necessarily type R, let L denote the left regular representation of S in $L^2(S/\Gamma)$.

It follows from (II.2) and the above reasoning that $\mathfrak{H}_\infty^\omega(L)$ can be identified with a subspace of $\Omega(M_c)$. In fact, since M is compact, it is easily seen that $\Omega(M_c) = \mathfrak{H}_\infty^\omega(L)$. It follows from (II.4) of [9] that every element of $\Omega(M_c)$ is left fixed by left translation by elements of the Green kernel K of S (recall that the Green kernel is the smallest analytic normal subgroup K for which S/K is type R). In particular, every element of $\Omega(M_c)$ is constant on K. If $\Omega(M_c)$ separates points of M, then $K \subset \Gamma$. Hence $K = \{e\}$ and S is type R.

(ii) Let $\pi: S_c \to M_c$ be the projection map and let $\Gamma' = \{z \in S_c \mid f(\pi(z)) = f(\pi(e)) \ \forall f \in \Omega(M_c)\}$. Γ' is a closed subgroup of S_c which contains Γ. We claim that Γ' is a complex subgroup. To see this, let $Z \in \mathfrak{L}(S_c)$ be such that $\exp tZ \in \Gamma'$ for all $t \in \mathbf{R}$.

If $f \in \Omega(M_c)$, then the map $w \to f(\pi(\exp wZ))$ of $\mathbf{C}$ into $\mathbf{C}$ is holomorphic and is constant on $\mathbf{R}$. Hence it is constant and, in particular $f(\pi(\exp itZ)) = f(\pi(e))$ for all $t \in \mathbf{R}$. Thus $iZ \in \mathfrak{L}(\Gamma')$, as claimed.

Similarly, it follows that Γ' is invariant under the canonical conjugation in S_c. Therefore $\mathfrak{L}(\Gamma')$ is the complexification of a real Lie subalgebra of S. Since $\Gamma' \cap S = \Gamma$ (since $\Omega(M_c)$ separates points of S/Γ), it follows that $\mathfrak{L}(\Gamma')$ is the complexification of $\mathfrak{L}(\Gamma)$ and hence that $(\Gamma')_0 = (\Gamma_0)_c$, as claimed. The rest of the proposition follows. Q.E.D.

Remarks. We know of no examples where $\Gamma' \neq \Gamma_c$. It would be interesting to know if such examples exist.

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